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§1.

The dimension of the space of the Hilbert cusp forms has been calculated in most of the cases, but not yet for the case of weight 1. In this report, we shall talk of some dimension formula for this remaining case under some restrictions. Fundamental references used here are the following:

[1] I. Y. Efrat : The Selberg trace formula for $PSL_2(\mathbb{R})^n$, Memoirs A. M. S., 1987.

[2] S. Gelbart and H. Jacquet : Forms of $GL(2)$ from the analytic point of view, Proc. Symposia in Pure Math., 33(1979).

[3] H. Shimizu : A remark on the Hilbert modular forms of weight 1, Math. Ann., 265(1983).

[4] P. Zograf : Selberg trace formula for the Hilbert modular group of a real quadratic alg. n. f., J. Soviet Math., 19(1982).

§2.

Firstly we shall introduce some notation:

$\mathfrak{H} = \{z : Im z > 0\}$, T : real torus

$\mathfrak{H}^2 = \{w = (z_1, z_2) : z_j \in \mathfrak{H} \ (j = 1,2)\}$

$$\tilde{\mathfrak{H}} = \{\tilde{z} = (z, \phi) : z \in \mathfrak{H}, \phi \in T\}$$
$$\tilde{\mathfrak{H}}^2 = \{\tilde{w} = (\tilde{z}_1, \tilde{z}_2) : \tilde{z}_j \in \tilde{\mathfrak{H}} (j = 1,2)\}$$
$$G' = SL(2, \mathbb{R}), \ \tilde{G}' = G' \times T, \ G'^2 = G' \times G', \ \tilde{G}'^2 = \tilde{G}' \times \tilde{G}'.$$

The operation of $\sigma = (\sigma_1, \sigma_2) \in \tilde{G}'^2$ on $\tilde{\mathfrak{H}}^2$ is represented as follows:

$$\sigma(\tilde{w}) = (\sigma_1\tilde{z}_1, \sigma_2\tilde{z}_2);$$
$$\sigma_j\tilde{z}_j = (g_j, \alpha_j)(z_j, \phi_j) = \left(\frac{a_j z_j + b_j}{c_j z_j + d_j}, \phi_j + \arg(c_j z_j + d_j) - \alpha_j\right)$$

for $\sigma_j = (g_j, \alpha_j)\left(g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in G', \alpha_j \in T\right)$ with $j = 1,2$.

The $\tilde{G}'^2$-invariant metric on $\tilde{\mathfrak{H}}^2$ is

$$ds^2 = \sum_{j=1}^{2} \left\{ \frac{dx_j + dy_j}{y_j^2} + \left(d\phi_j - \frac{dx_j}{2y_j}\right)^2 \right\},$$

and the $\tilde{G}'^2$-invariant measure $d\tilde{w}$ associated to ds^2 is given by

$$d\tilde{w} = \prod_{j=1}^{2} \frac{dx_j dy_j d\phi_j}{y_j^2},$$

where $z_j = x_j + \sqrt{-1}\, y_j$ $(j = 1,2)$. The ring of $\tilde{G}'^2$-invariant differential operators on $\tilde{\mathfrak{H}}^2$ is generated by

$$\frac{\partial}{\partial\phi_j}, \quad \tilde{\Delta}_j = y_j^2\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) + \frac{5}{4}\frac{\partial^2}{\partial\phi_j^2} + y_j \frac{\partial}{\partial\phi_j}\frac{\partial}{\partial x_j} \quad (j = 1,2).$$

A discrete subgr. Γ of G'^2 is said to be irreducible if Γ is not commensurable with a direct product $\Gamma_1 \times \Gamma_2$, where $\Gamma_1 \subset G_1$, $\Gamma_2 \subset G_2$ are discrete, G_1 and G_2 are not trivial, and $G'^2 = G_1 \times G_2$. Let F be a real quadratic field and O_F be the ring of integers of F. We put

$$SL(2, O_F) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, \cdots, d \in O_F\right\}.$$

Then the Hilbert modular group associated to F is

$$\Gamma_F = \{(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, O_F)\},$$

where $a', \cdots, d'$ denotes conjugate of $a, \cdots, d$ respectively. It is well known that Γ_F is discrete and an irreducible subgr., and that its number of cusps equals the class number $h(F)$ of F.

Now, for $\gamma = (\gamma_1, \gamma_2) = (\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}) \in \Gamma$ and $w \in \mathfrak{H}^2$, we put

$$j(\gamma, w) = \{(c_1 z_1 + d_1)(c_2 z_2 + d_2)\}^{-1}.$$

We say that a function $F(w)$ defined on $\mathfrak{H}^2$ is a Hilbert cusp form of weight 1 for Γ if

1) $F(w)$ is holomorphic on $\mathfrak{H}^2$,

2) $F(\gamma(w)) = j(\gamma, w)^{-1} F(w)$ for all $\gamma \in \Gamma$,

3) At every parabolic point κ of Γ, a constant term in the Fourier expansion of $F(w)$ at κ vanishes.

We denote by $\mathfrak{S}(1, \Gamma)$ the space of Hilbert cusp forms of weight 1 for Γ and put

$$d_1 = \dim \mathfrak{S}(1, \Gamma).$$

In the following we shall calculate the dimension d_1 for some Γ.

Here we give a remark:

Remark 1. For the case of n complex variables and weight $r = (r_1, \cdots, r_n)$, the dimension formulas have been calculated in the following papers:

$\forall r_j > 2$, even : H. Shimizu(1963),

$\forall r_j = 2$: K.-B. Merz(1971),

$\forall r_j = 2$, $n = 2$: F. Hirzebruch(1973) by alg. geo. method,

$\forall r_j \geqq 2$: H. Ishikawa(1979),

$\forall r_j = 1$, $n = 2$: Our case.

3. Fundamental lemma

We denote by $\mathfrak{M}_\Gamma((k_1, \lambda_1), (k_2, \lambda_2))$ the set of all functions $f(\tilde{w})$ satisfying the following conditions:

(i) $f(\tilde{w}) \in L^2(\Gamma\backslash\tilde{\mathfrak{H}}^2)$, i.e.,

$$\int_{\Gamma\backslash\tilde{\mathfrak{H}}^2} |f|d\tilde{w} < \infty \quad \text{and} \quad f(\gamma(\tilde{w})) = f(\tilde{w}) \quad \text{for all} \quad \gamma \in \Gamma,$$

(ii) For $j = 1, 2$

(3.1) $\tilde{\Delta}_j f(\tilde{w}) = \lambda_j f(\tilde{w})$,

(3.2) $\dfrac{\partial}{\partial\phi_j} f(\tilde{w}) = -\sqrt{-1}\, k_j f(\tilde{w})$.

Then the following lemma holds:

Lemma. If Γ is arithmetic, then

$$d_1 = \dim \mathfrak{M}_\Gamma\left((1, -\tfrac{3}{2}), (1, -\tfrac{3}{2})\right).$$

Out line of proof : Let $\mathbb{A}$ be the adele ring of the real quadratic field F. Let G_F be $GL(2, F)$ viewed as an alg. gr. over F and $G_{\mathbb{A}}$ the adelization of G_F. In the following Hilbert modular forms may be viewed as automorphic forms on $G_{\mathbb{A}}$.

Firstly we put

$$G_{\mathbb{A}} = G_{\mathbb{A}_f} \times G_{\mathbb{A}\infty}$$

K_f : open compact subgr. of $G_{\mathbb{A}_f}$

$$G_\infty^+ = GL(2, \mathbb{R})^+ \times GL(2, \mathbb{R})^+$$

$$\Gamma = G_F \cap (K_f \times G_\infty^+).$$

Then Γ is a discrete subgr. of G_∞^+; and we have

$$G_{\mathbb{A}} = \bigcup_{j=1}^{h} G_F x_j (K_f \times G_\infty^+) \quad \text{(disjoint)}.$$

Let $f(\tilde{w})$ be a function in $\mathfrak{M}_\Gamma\big((k_1, \lambda_1), (k_2, \lambda_2)\big)$. Put

$$f(\tilde{w}) = e^{-\sqrt{-1}(k_1\phi_1+k_2\phi_2)}y_1^{\frac{k_1}{2}}y_2^{\frac{k_2}{2}}F(z_1, z_2).$$

Then the Γ-invariance of $f(\tilde{w})$ is equivalent to a transformation law for $F(z_1, z_2)$:

$$F(\gamma_1 z_1, \gamma_2 z_2) = \left[\frac{c_1z_1 + d_1}{(\det\gamma_1)^{\frac{1}{2}}}\right]^{k_1}\left[\frac{c_2z_2 + d_2}{(\det\gamma_2)^{\frac{1}{2}}}\right]^{k_2}F(z_1, z_2)$$

for all $\gamma = (\gamma_1, \gamma_2) = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1\end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2\end{pmatrix}\right) \in \Gamma$.

Now we put

$$\Gamma_j = G_F \cap (x_jK_fx_j^{-1} \times G_\infty^+) \quad (1 \leqq j \leqq h);$$

and denote by M_j the set of functions satisfying the conditions (i) and (ii) for Γ_j. Moreover denote by M the space of all ψ on $G_{\mathbb{A}}$ satisfying the following conditions.

(1) $\psi(\alpha x k_f k_\infty t) = e^{\sqrt{-1}(k_1\theta_1+k_2\theta_2)}\psi(x)$,

where $\alpha \in G_F$, $k_f \in K_f$, $k_\infty = \big(k(\theta_1), k(\theta_2)\big) \in K_\infty$,

$k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$ and $t \in Z_\infty^+ = \left\{\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_1\end{pmatrix}, \begin{pmatrix} t_2 & 0 \\ 0 & t_2\end{pmatrix}\right) : t_1, t_2 > 0\right\}$;

(2) $\tilde{\Delta}_j\psi = \lambda_j\psi (j = 1,2)$; (3) $\displaystyle\int_{Z_\infty^+ G_F\backslash G_{\mathbb{A}}} |\psi(g)|^2 dg < \infty$.

For $x \in G_{\mathbb{A}}$, we put $x = \alpha x_j k g (\alpha \in G_F,\ k \in K_f,\ g \in G_\infty^+)$ and

$$\psi(x) = f_j(gz_0) \quad \big(z_0 = \big((\sqrt{-1}, 0), (\sqrt{-1}, 0)\big)\big).$$

Then $(f_1, \cdots f_h) \to \psi$ gives an isomorphism of $M_1 \times \cdots \times M_h$ onto M.

Let Z be the center of $G = GL(2)$ and ω character of $Z_{\mathbb{A}}/Z_F(Z_{\mathbb{A}} \cap K_f)Z_\infty^+$. Then we put

$$L^2(\omega, G) = \left\{\psi : G_{\mathbb{A}} \to \mathbf{C} \;\middle|\; \begin{array}{l}\psi(a\alpha x) = \omega(a)\psi(x) \quad \text{for all } \alpha \in G_F \\ \displaystyle\int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} |\psi(x)|^2 dx < \infty\end{array}\right\}.$$

The space $\bigoplus_{\omega} L^2(\omega, G)$ contains M. Now we have that $L^2(\omega, G)$ decomposes as a direct sum (Gelbart-Jacquet [2])

$$L^2(\omega, G) = L^2{}_{\text{cusp}}(\omega, G) \oplus L^2{}_{\text{sp}}(\omega, G) \oplus L^2{}_{\text{cont}}(\omega, G),$$

where

$$L^2{}_{\text{cusp}}(\omega, G) = \{\psi \in L^2(\omega, G) : \int_{N_{\mathbb{A}}/N_F} \psi(ng)dn = 0 \text{ for all } g \in G_{\mathbb{A}}\}$$

with $N = \{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\}$, $L^2{}_{\text{sp}}$ is the space spanned by $\chi(\det g)$ with a character χ on $\mathbb{A}^\times/F^\times$ such that $\chi^2 = \omega$.

<u>Lemma 1.1</u> The space $L^2\text{cont}(\omega, G)$ does not contain the eigenfunctions of Laplacian $\tilde{\Delta}_j - \frac{5}{4}\frac{\partial^2}{\partial\phi_j}$ $(j = 1,2)$.

<u>Lemma 1.2</u> If $f \in M$, then $F(z_1, z_2)$ is holomorphic.

§4. <u>Modified trace formula</u>

We put $\lambda_j = (k_j, \lambda_j)(j = 1,2)$. For every invariant integral operator with a kernel function $k(\tilde{w} ; \tilde{v})$ on $\mathfrak{M}_\Gamma(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$, we have Selberg's trace formula of compact type:

$$\int_{\tilde{\mathfrak{H}}^2} k(\tilde{w} ; \tilde{v})f(\tilde{v})d\tilde{v} = \tilde{h}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)f(\tilde{w})$$

for $f \in \mathfrak{M}_\Gamma(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$. We put

$$K(\tilde{w} ; \tilde{v}) = \sum_{\gamma \in \Gamma} k(\tilde{w} ; \gamma\tilde{v});$$

then we have

$$\int_{\Gamma\backslash\tilde{\mathfrak{H}}^2} (\tilde{w} ; \tilde{v})f(\tilde{v})d\tilde{v} = \tilde{h}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)f(\tilde{w}).$$

Denote by $\Gamma(\gamma)$ the centralizer of γ in Γ, and put $\tilde{F}_{\Gamma(\gamma)} = \Gamma(\gamma)\backslash\tilde{\mathfrak{H}}^2$. Then it is easy to see that

$$\int_{\Gamma\backslash\tilde{\mathfrak{H}}^2} K(\tilde{w} ; \tilde{w})d\tilde{w} = \sum_{\{\gamma\}} \sum_{\sigma \in \Gamma/\Gamma(\gamma)} \int_{\Gamma\backslash\tilde{\mathfrak{H}}^2} k(\tilde{w} ; \sigma^{-1}\gamma\sigma\tilde{w})d\tilde{w}$$

$$= \sum_{\{\gamma\}} \int_{F_{\Gamma(\gamma)}} k(\tilde{w} \,;\, \gamma\tilde{w})d\tilde{w},$$

where the sum $\{\gamma\}$ is taken over the distinct conjugacy classes of Γ.

Since our Γ is non compact type its spectrum has a continuous part, and the continuous spectrum can be described by a family of Eisenstein series. Using the Eisenstein series, we shall construct in after section a new kernel H_δ ; then $K - H_\delta$ is now a Hilbert-Schmidt kernel. Therefore we have the following modified trace formula

$$[*] \quad \sum_{j=1}^{\infty} \tilde{h}(\boldsymbol{\lambda}_1^{(j)}, \boldsymbol{\lambda}_2^{(j)}) = \int_{\Gamma\backslash\tilde{\mathfrak{H}}^2} \{K(\tilde{w} \,;\, \tilde{w}) - H_\delta(\tilde{w} \,;\, \tilde{w})\}d\tilde{w}.$$

Now we consider the following invariant integral operator defined by

$$\omega_\delta(\tilde{w} \,;\, \tilde{v}) = \prod_{j=1}^{2} \{ \left| \frac{(y_j y_j')^{\frac{1}{2}}}{(z_j - \bar{z}_j')/2\sqrt{-1}} \right|^\delta \frac{(y_j y_j')^{\frac{1}{2}}}{(z_j - \bar{z}_j')/2\sqrt{-1}} e^{-\sqrt{-1}(\phi_j - \phi_j')} \},$$

where $\tilde{v} = (\tilde{z}_1', \tilde{z}_2')$ and $\delta > 1$. The integral operator ω_δ vanishes on $\mathfrak{M}_\Gamma(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ for all $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ except $k_1 = k_2 = 1$. we denote by

$$\mu_{\alpha\beta} = ((1, \lambda^{(\alpha)}), (1, \lambda^{(\beta)})), \quad \alpha, \beta \geqq 2$$

$$\mu_{11} = ((1. -\tfrac{3}{2}), (1, -\tfrac{3}{2})), \quad \lambda^{(1)} = -\tfrac{3}{2}$$

the discrete part of spectra, and we put

$$d_{\alpha\beta} = \dim \mathfrak{M}_\Gamma(\mu_{\alpha\beta}).$$

Then the left-hand side of [*] implies

$$\sum_{j=1}^{\infty} \tilde{h}(\boldsymbol{\lambda}_1^{(j)}, \boldsymbol{\lambda}_2^{(j)}) = \sum_{\alpha\,\beta=1}^{\infty} d_{\alpha\beta}\Lambda_{\alpha\beta},$$

where $\Lambda_{\alpha\beta}$ denotes the eigenvalue of ω_δ in $\mathfrak{M}_\Gamma(\mu_{\alpha\beta})$. For the eigenvalue $\Lambda_{\alpha\beta}$, we have

$$\Lambda_{\alpha\beta} = \left\{2^{2+\delta}\pi\frac{\Gamma(\frac{1}{2})\Gamma(\frac{\delta+1}{2})}{\Gamma(\delta)\Gamma(1+\frac{\delta}{2})}\right\}^2 \Gamma(\tfrac{\delta}{2}+\sqrt{-1}u_\alpha)\Gamma(\tfrac{\delta}{2}-\sqrt{-1}u_\alpha)\Gamma(\tfrac{\delta}{2}+\sqrt{-1}u_\beta)\Gamma(\tfrac{\delta}{2}-\sqrt{-1}u_\beta),$$

where $\lambda^{(\ell)} = r_\ell(r_\ell - 1) - \frac{5}{4}$ and $r_\ell = \frac{1}{2} + \sqrt{-1}\, u_\ell$ with $\ell = \alpha, \beta$. In general, it is known that the series $\sum_{\alpha,\beta=1}^{\infty} d_{\alpha\beta}\Lambda_{\alpha\beta}$ is absolutely convergent for $\delta > 1$, and by the Stirling formula, we see that the above series is absolutely and uniformly convergent for all bounded δ except $\delta = \pm(2r_\alpha - 1)$, $\pm(2r_\beta - 1)$. Note that

$$\delta = (0,\ 0) \Longleftrightarrow \left(\lambda^{(\alpha)},\lambda^{(\beta)}\right) = \left(\lambda^{(1)},\lambda^{(1)}\right).$$

§5. Compact contribution

1) $\gamma = 1_2$.

$$\int_{\tilde{F}_{\Gamma(\gamma)}} d\tilde{\omega} = \int_{\tilde{F}_\Gamma} d\tilde{\omega} < \infty \quad (\tilde{F}_\Gamma = \Gamma\backslash\mathfrak{H}^2).$$

2) γ : totally elliptic.

We put $\gamma = (\gamma_1, \gamma_2)$, and let ζ_j, $\overline{\zeta}_j$ be the eigenvalues of γ_j. Consider a linear transformation that maps $\mathfrak{H}^2$ into the product of the 2 unit circles, and a fixed point of γ to its origin. Then

$$\frac{\gamma_j z_j - \rho_j}{\gamma_j z_j - \overline{\rho}_j} = \frac{\zeta_j}{\overline{\zeta}_j}\,\frac{z_j - \rho_j}{z_j - \overline{\rho}_j} \qquad (j = 1,2),$$

where $\rho = (\rho_1, \rho_2) \in \mathfrak{H}^2$ is the fixed point of γ. By a simple calculation, we have the following contribution from this part:

$$\lim_{\delta\to 0} \delta^2 J(\gamma) = \frac{(8\pi^2)^2}{[\Gamma(\gamma) : Z(\Gamma)]} \sum_{\{\gamma\}:\text{elliptic}} \prod_{j=1}^{2} \frac{\overline{\zeta}_j}{1 - \overline{\zeta}_j^2},$$

where $Z(\Gamma)$ denotes the center of Γ.

3) γ : totally hyperbolic st. no fixed point of γ is a parabolic point of Γ.

The contribution from this part is essential in the case of weight 1 : We put $\gamma = (\gamma_1, \gamma_2)$. Then there exists some g_j in G' such that

$$g_j^{-1}\gamma_j g_j = \begin{pmatrix} \lambda_{0j} & 0 \\ 0 & \lambda_{0j}^{-1} \end{pmatrix}, \quad |\lambda_{0j}| > 1.$$

Let $\{\gamma_1, \gamma_2\}$ be a system of generators of $\Gamma(\gamma)$ and $\lambda_0^{(\ell)} = (\lambda_{01}^{(\ell)}, \lambda_{02}^{(\ell)})$ $(|\lambda_0^{(\ell)}| > 1,\ \ell = 1,2)$ denotes an eigenvalue of γ_ℓ respectively. Writing $z_j = \rho_j e^{\sqrt{-1}\theta_j}$ and $\log\rho_j = u_1\log\lambda_{0j}^{(1)} + u_2\log\lambda_{0j}^{(2)}$ with $u_\ell \in \mathbb{R}$, the set of $z = (z_1, z_2)$ such that

$$0 < u_\ell < 1 \quad (\ell = 1,2),\ 0 < \theta_j < \pi \quad (j = 1,2)$$

forms a fundamental domain of $\Gamma(\gamma)$ in $\mathfrak{H}^2$.;
and the contribution from this part is as follows:

$$J(\gamma) = 2^{6+2\delta}\pi^3\left\{\frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta+2}{2})}\right\}^2 \sum_{\{\gamma\}\in\mathfrak{S}} \prod_{j=1}^{2} \frac{\operatorname{sgn}\lambda_j \cdot \mu(\gamma)}{|\lambda_j + \lambda_j^{-1}|^\delta |\lambda_j - \lambda_j^{-1}|}$$

where $\mathfrak{S}$ denotes a complete system of totally hyp. conjugacy classes in Γ such that none of its fixed pts. is a parabolic element of Γ, $\lambda = (\lambda_1, \lambda_2)$ an eigenvalue of γ and $\mu(\gamma) = \det(\log|\lambda_{0j}^{(\ell)}|)_{j,\ell=1,2}$. Multiply $J(\gamma)$ by δ^2 and tend δ to zero, then the limit is expressed by $(8\pi^2)^2\operatorname{Res}_{\delta=0} Z(\delta)$, where

$$Z(\delta) = \sum_{\{\gamma\}} \prod_{j=1}^{2} \frac{\operatorname{sgn}\lambda_j \cdot \mu(\gamma)}{|\lambda_j + \lambda_j^{-1}|^\delta |\lambda_j - \lambda_j^{-1}|}$$

By the trace formula [*], the function $Z(\delta)$ extends to a meromorphic function on the whole δ-plane and has a double pole at $\delta = 0$ whose residue will appear in d_1.

4) γ : mixed

The contribution from this part also vanishes.

§6. Eisenstein series attached to ∞

For the sake of simplicity, we shall assume that F_Γ has only one cusp, i.e., the cusp is at $\infty = (\infty, \infty)$.

Let $s \in \mathbf{C}$, $m \in \mathbf{Z}$ and Γ_∞ be the stabilizer of ∞ in Γ. Then the Eisenstein series attached to ∞ is defined by

$$E(\tilde{w};\ s,\ m) = \sum_{\substack{M\ \Gamma_\infty\ \Gamma \\ M=(M_1,M_2)}} \{y(M_1(z_1))y(M_2(z_2))\}^s$$

$$\times\ e^{-\sqrt{-1}\{\phi_1+\phi_2+\arg(cz_1+d)(c'z_2+d')\}}$$

$$\times\ \{y(M_1(z_1))\}^{\frac{\pi\sqrt{-1}m}{2\log\varepsilon}}\{y(M_2(z_2))\}^{-\frac{\pi\sqrt{-1}m}{2\log\varepsilon}},$$

where ε denotes the fundamental unit (> 1) of F, $y(z) = Im z$, $M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $M_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. $E(\tilde{w};\ s,\ m)$ converges absolutely for $\mathrm{Re}\, s > 1$, and has the following properties:

$$E(\gamma\tilde{w};\ s,\ m) = E(\tilde{w};\ s,\ m) \quad \text{for} \quad \gamma \in \Gamma;$$

$$\frac{\partial}{\partial\phi_j} E = -\sqrt{-1}\, E,\quad \tilde{\Delta}_j\, E = \lambda_j\, E \quad (j = 1,2),$$

where

$$\begin{cases} \lambda_1 = \left(s + \dfrac{\pi\sqrt{-1}m}{2\log\varepsilon}\right)\left(s + \dfrac{\pi\sqrt{-1}m}{2\log\varepsilon} - 1\right) - \dfrac{5}{4}, \\[2ex] \lambda_2 = \left(s - \dfrac{\pi\sqrt{-1}m}{2\log\varepsilon}\right)\left(s - \dfrac{\pi\sqrt{-1}m}{2\log\varepsilon} - 1\right) - \dfrac{5}{4}. \end{cases}$$

We put

$$E(w;\ s,\ m) = e^{\sqrt{-1}(\phi_1+\phi_2)}E(\tilde{w};\ s,\ m).$$

Then the series $E(w;\ s,\ m)$ is invariant under the action of the lattice O_F, and therefore has a Fourier expansion of the form

$$E(w;\, s,\, m) = \sum_{\substack{\ell \in O_F^* \\ \ell=(\ell_1,\ell_2)}} a_\ell(y;\, s,\, m) e^{2\sqrt{-1}\pi<\ell,x>},$$

where $<\ell, x> = \ell_1 x_1 + \ell_2 x_2$, and O_F^* is the dual lattice of O_F, i.e., $O_F^* = \{\alpha \in F : \mathrm{tr}(\alpha O_F) \subset \mathbf{Z}\}$. The constant term $a_0(y;\, s,\, m)$ is given by

$$a_0(y;s,m) = y_1^{s+\frac{\pi\sqrt{-1}m}{2\log\varepsilon}} y_2^{s-\frac{\pi\sqrt{-1}m}{2\log\varepsilon}} - \psi(s,m) y_1^{1-s-\frac{\pi\sqrt{-1}m}{2\log\varepsilon}} y_2^{1-s+\frac{\pi\sqrt{-1}m}{2\log\varepsilon}},$$

where

$$\psi(s,m) = \frac{\pi}{\sqrt{D}} \frac{\Gamma(s + \frac{\pi\sqrt{-1}}{2\log\varepsilon})\Gamma(s - \frac{\pi\sqrt{-1}}{2\log\varepsilon})}{\Gamma(s + \frac{\pi\sqrt{-1}m}{2\log\varepsilon} + \frac{1}{2})\Gamma(s - \frac{\pi\sqrt{-1}m}{2\log\varepsilon} + \frac{1}{2})} \cdot \frac{L(2s-1,\, -m)}{L(2s,\, -m)},$$

D : disc. of F,

$$\xi_{-m}(c) = \left| \frac{c}{c'} \right|^{-\frac{\pi\sqrt{-1}m}{\log\varepsilon}} \quad : \text{Grössencharacter of } F,$$

$$L(s,\, -m) = \sum_{\substack{(c):\text{ideal in } O_F \\ c\neq 0}} \mathrm{sgn}(cc')\xi_{-m}(c)|N(c)|^{-s}.$$

Now, by using the analytic continuation of the Eisenstein series $E(\tilde{w};\, s,\, m)$ as a function of s for $s = \frac{1}{2} + \sqrt{-1}r$ (Efrat [1]), we put

$$H_\delta(\tilde{w}\,;\, \tilde{v}) = \frac{1}{16\pi\sqrt{D}\log\varepsilon} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{h}_\delta(r + \frac{\pi m}{2\log\varepsilon},\, r - \frac{\pi m}{2\log\varepsilon})$$

$$\times E(\tilde{w};\, \tfrac{1}{2} + \sqrt{-1}r,\, m) E(\tilde{v};\, \tfrac{1}{2} - \sqrt{-1}r,\, -m)\,dr,$$

where

$$\tilde{h}_\delta(r_1,r_2) = \left\{2^{2+\delta}\pi \frac{\Gamma(\frac{1}{2})\Gamma(\frac{\delta+1}{2})}{\Gamma(\delta)\Gamma(1+\frac{\delta}{2})}\right\}^2 \Gamma(\tfrac{\delta}{2} + \sqrt{-1}r_1)$$

$$\times \Gamma(\tfrac{\delta}{2} - \sqrt{-1}r_1)\Gamma(\tfrac{\delta}{2} + \sqrt{-1}r_2)\Gamma(\tfrac{\delta}{2} - \sqrt{-1}r_2).$$

Then the integral operator $K - H_\delta$ is now completely continuous on $L^2(\Gamma\backslash\mathfrak{H}^2)$ and has all discrete spectra of K.

§7. The trace at the cusp

5) γ : totally parabolic.

We have the following which is obtained in a similar way as in the elliptic modular case:

$$\lim_{\delta\to 0}\lim_{Y\to 0} \delta^2\{J(\infty) - 2\log\varepsilon\cdot g_\delta(0, 0)\log Y\} = 0.$$

Therefore the contribution from parabolic classes to d_1 vanishes.

6) γ : hyp.-parabolic.

γ is conjugate in Γ to $\gamma_{m,\alpha}$:

$$\gamma \underset{\Gamma}{\sim} \gamma_{m,\alpha} = \begin{pmatrix}\varepsilon^m & \alpha\\ 0 & \varepsilon^{-m}\end{pmatrix}, \ (m \in \mathbb{Z},\ m \neq 0,\ \alpha \in \mathcal{O}_F).$$

The common fixed pts. of every element in $\Gamma(\gamma)$ is

$$\{\infty, \frac{\alpha}{\varepsilon^{-m} - \varepsilon^m}\},$$

and there exists a $\tau \in \Gamma$ such that

$$\tau : \frac{\alpha}{\varepsilon^{-m} - \varepsilon^m} \longrightarrow \infty.$$

We denote by $F_{\Gamma(\gamma_{m,\alpha})}$ a fundamental domain of $\Gamma(\gamma_{m,\alpha})$. Take $Y \gg 0$ and we put

$$F^*_{\Gamma(\gamma_{m,\alpha})} = \{w = (z_1, z_2) \in F_{\Gamma(\gamma_{m,\alpha})} : y_1 y_2 \leq Y,\ y_1' y_2' \leq Y\},$$

where $\tau w = w' = (z_1', z_2')$; moreover we put

$$J^*(\gamma) = 4\pi^2 \sum_{\{\gamma\}} \int_{F^*_{\Gamma(\gamma)}} \omega_\delta(w\,;\gamma w)dw.$$

Then the contribution from this part is

$$\lim_{\delta\to 0}\lim_{Y\to\infty} \delta^2\{J^*(\gamma) - 4\log\varepsilon \sum_{m=1}^{\infty} \frac{g_\delta(2m\log\varepsilon, 2m\log\varepsilon')}{|(\varepsilon^m-\varepsilon^{-m})(\varepsilon'^m-\varepsilon'^{-m})|} \cdot \log Y\} = 0,$$

where

$$g_\delta(u_1,u_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{h}_\delta(r_1,r_2)e^{-\sqrt{-1}(r_1u_1+r_2u_2)}dr_1dr_2.$$

7) tr H_δ.

By using the Maass-Selberg relation (Efrat [1]), the following contribution may be obtained in a way similar to the proof of elliptic modular case:

$$-\frac{1}{4}\tilde{h}_\delta(0,\ 0)\psi(\tfrac{1}{2},\ 0).$$

Therefore

$$\lim_{\delta\to 0} \delta^2\{-\frac{1}{4}\tilde{h}_\delta(0,\ 0)\psi(\tfrac{1}{2},\ 0)\} = -(8\pi^2)^2\psi(\tfrac{1}{2},\ 0).$$

We note that $\psi(\frac{1}{2}, 0) = \pm 1$.

Summing up the above results we have the following final form of $d_1 = d_{11}$:

$$\boxed{d_1 = \frac{1}{4}\sum_{\{\gamma\}} \frac{1}{[\Gamma(\gamma):Z(\Gamma)]}\prod_{j=1}^{2}\frac{\overline{\zeta}}{1-\overline{\zeta}_j^2} + \frac{1}{4}\operatorname*{Res}_{\delta=0} Z(\delta) - \frac{1}{4}\psi(\tfrac{1}{2},\ 0).}$$

Here we give a final remark:

<u>Remark 2.</u> For the general case of several cusps, the contribution of the parabolic elements is simply computed separately at each cusp. But that of the hyp.-parabolic elements is more complicated, since a typical element fixes two different cusps.